# TIGHT BOUNDS ON THE CARATHÉODORY AND EXCHANGE NUMBERS IN Δ-CONVEXITY

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Abstract. For a graph $G$, a set $S \subseteq V(G)$ is $\Delta$-convex if every vertex that forms a triangle with two vertices of $S$ already belongs to $S$. The associated convexity gives rise to the Carathéodory number $c_\Delta(G)$, namely, the maximum cardinality of a Carathéodory independent set. A recent paper of Anand, Anil, Changat, Narasimha-Shenoi and Ramla claimed that if $G$ has $k$ triangles, then $c_\Delta(G) \leq k+1$, but the proof given there is incomplete. In this note, we provide a new counting proof of this bound. The proof is based on a partition of the triangles of $G$ according to the connected components of the graph induced by a Carathéodory independent set. This leads to a short structural argument, together with an induction step for one exceptional configuration. We also indicate how the same method should extend to exchange numbers and to extremal characterizations.

## 1. Introduction

Convexity in graphs has been studied from several perspectives, with several notions arising from different combinatorial structures. Classical invariants of convexity spaces such as the Carathéodory number, Helly number, Radon number, and exchange number have natural graph-theoretic analogues, and have been investigated in a variety of settings. These include geodesic convexity, monophonic convexity, $P_3$-convexity, and triangle-path convexity; see, for instance, [3, 4, 2] for background.

In this work, we consider $\Delta$-convexity. Let $G$ be a finite simple graph. A set $S \subseteq V(G)$ is said to be $\Delta$-convex if there is no vertex in $V(G) \setminus S$ that forms a triangle with two vertices of $S$. Equivalently, $S$ is $\Delta$-convex if whenever a vertex forms a triangle with two vertices of $S$, that vertex already belongs to $S$. The $\Delta$-convex hull of a set $A \subseteq V(G)$, denoted by $\mathrm{Hull}(A)$, is the smallest $\Delta$-convex set containing $A$.

The $\Delta$-convexity has been studied recently in connection with structural and algorithmic questions. In particular, Anand, Anil, Changat, Narasimha-Shenoi, and Ramla [1] investigated the Carathéodory number and the exchange number in this setting, extending earlier work on other graph convexities. The Carathéodory number $c_\Delta(G)$ is defined as the maximum size of a Carathéodory independent set, where a nonempty set $A \subseteq V(G)$ is said to be Carathéodory independent if there exists a vertex

$$p \in \mathrm{Hull}(A) \setminus \bigcup_{a \in A} \mathrm{Hull}(A \setminus \{a\}).$$

One of the basic results stated in [1] is that if $G$ has $k$ triangles, then

$$c_\Delta(G) \leq k + 1.$$

This bound is natural, and examples show that it is tight. Our aim in this note is to provide a correct and transparent proof of this inequality.

The proof we present is combinatorial in nature. Given a Carathéodory independent set $S$, we partition the triangles of $G$ according to the connected components of the induced subgraph $G[S]$, and then count them componentwise. This leads to a direct argument in all but one exceptional configuration, which can be handled either by a reduction step or by identifying an additional triangle. The resulting proof isolates the precise structural reason behind the bound.

## 2. On the earlier proof

We briefly indicate why a new proof is needed.

In [1, Theorem 2.3], the authors state that if $G$ has $k$ triangles, then $c_\Delta(G) \leq k+1$. The proof correctly establishes two key structural facts about Carathéodory independent sets:

(1) every vertex of such a set lies on a triangle;

(2) no triangle is entirely contained in the set.

These observations form the natural starting point for any counting argument.

However, the final step of the proof requires showing that if a set $S$ is sufficiently large, then its convex hull is covered by the union of the convex hulls of its proper subsets obtained by deleting one element at a time. The argument given in [1] reduces this to the existence of multiple triangles meeting $S$ in two vertices, but does not establish the required covering relation for the convex hulls. In particular, the passage from local triangle structure to a global convex hull identity is not entirely straightforward, and requires further justification.

Since the bound itself is correct and sharp, it is natural to seek a replacement proof that avoids this issue. The argument given in the next section proceeds instead by assigning triangles to the connected components of $G[S]$ and counting them directly. This avoids the need for any global convex hull decomposition, and leads to a short and self-contained proof.

## 3. Preliminaries

Throughout, all graphs are finite, simple, undirected and connected. We write $k(G)$ for the number of triangles in $G$.

We recall two basic structural properties of Carathéodory independent sets in $\Delta$-convexity.

**Lemma 1.** *Let $S \subseteq V(G)$ be a Carathéodory independent set. Then:*

(1) *every vertex of $S$ lies in a triangle of $G$;*

(2) *no triangle of $G$ is contained in $S$.*

*Proof.* If $u \in S$ does not lie in any triangle, then

$$\mathrm{Hull}(S) = \mathrm{Hull}(S \setminus \{u\}) \cup \{u\},$$

so $S$ is Carathéodory dependent, a contradiction.

If $u, v, w \in S$ form a triangle, then

$$\mathrm{Hull}(S) = \mathrm{Hull}(S \setminus \{u\}) = \mathrm{Hull}(S \setminus \{v\}) = \mathrm{Hull}(S \setminus \{w\}),$$

again a contradiction. □

In proving upper bounds on $c_\Delta(G)$, we may assume without loss of generality that every edge of $G$ lies in a triangle. Indeed, removing an edge that is not contained in any triangle does not affect the $\Delta$-convexity, and hence does not affect Carathéodory independence.

Let $S \subseteq V(G)$ be a Carathéodory independent set, and consider the induced subgraph $G[S]$. For each connected component $H$ of $G[S]$, define

$$\mathcal{T}_H := \{T : T \text{ is a triangle of } G \text{ with } V(T) \cap V(H) \neq \emptyset\}.$$

**Lemma 2.** *If $H$ and $H'$ are distinct connected components of $G[S]$, then*

$$\mathcal{T}_H \cap \mathcal{T}_{H'} = \emptyset.$$

*Proof.* Let $T$ be a triangle of $G$ that meets $S$. By Lemma 1(2), $T$ cannot be entirely contained in $S$, so $|V(T) \cap S| \in \{1, 2\}$. If $|V(T) \cap S| = 2$, those two vertices are adjacent and hence lie in the same connected component of $G[S]$. If $|V(T) \cap S| = 1$, the vertex lies in a unique component. Thus $T$ is associated to exactly one component. □

The next lemma records the key counting principle.

**Lemma 3.** *Let $H$ be a connected component of $G[S]$.*

(1) *If $H$ is an isolated vertex, then $|\mathcal{T}_H| \geq |V(H)|$.*

(2) *If $H$ is nonempty and not acyclic, then $|\mathcal{T}_H| \geq |V(H)|$.*

(3) *If $H$ is a tree, then $|\mathcal{T}_H| \geq |V(H)| - 1$, with equality only if:*

   (a) *every edge of $H$ is the base of a unique triangle, and*

   (b) *there is no triangle having exactly one vertex in $H$.*

*Proof.* Since every edge of $G$ lies in a triangle, each edge of $H$ is the base of at least one triangle in $\mathcal{T}_H$.

If $H$ is an isolated vertex $u$, then $u$ lies in a triangle by Lemma 1(1), so $|\mathcal{T}_H| \geq 1$.

If $H$ is nonempty and contains a cycle, then

$$|\mathcal{T}_H| \geq |E(H)| \geq |V(H)|.$$

If $H$ is a tree, then

$$|\mathcal{T}_H| \geq |E(H)| = |V(H)| - 1.$$

If some edge of $H$ is the base of more than one triangle, or if there exists a triangle meeting $H$ in exactly one vertex, then one obtains at least one additional triangle, giving $|\mathcal{T}_H| \geq |V(H)|$. □

## 4. Main theorem

**Theorem 4.** *If $G$ has $k$ triangles, then*

$$c_\Delta(G) \leq k + 1.$$

*Proof.* We argue by induction on $k = k(G)$.

If $k = 0$, then $G$ is triangle-free. Hence no vertex can be generated from a set of size at least 2, and every such set is Carathéodory dependent. Therefore $c_\Delta(G) = 1 = k + 1$.

If $k = 1$, then any Carathéodory independent set has size at most 2, while any two vertices of the unique triangle form a Carathéodory independent set. Thus $c_\Delta(G) = 2 = k + 1$.

Now let $k \geq 2$, and suppose the result holds for all graphs with fewer than $k$ triangles. Let $S$ be a Carathéodory independent set in $G$. We may assume, as noted earlier, that every edge of $G$ lies in a triangle.

Let $H_1, \dots, H_m$ be the connected components of $G[S]$. By Lemma 2, the sets

$$\mathcal{T}_{H_1}, \dots, \mathcal{T}_{H_m}$$

are pairwise disjoint.

For each component $H_i$, Lemma 3 shows that

$$|\mathcal{T}_{H_i}| \geq |V(H_i)|$$

unless $H_i$ is a tree. For a tree component, Lemma 3 shows that the same inequality fails only in the exceptional situation where every edge of $H_i$ is the base of a unique triangle and there is no triangle having exactly one vertex in $H_i$.

If such an exceptional component $H_i$ consists of a single edge $uv$, then we perform an inductive reduction as follows: if $w$ is the apex of the unique triangle containing the edge $uv$, then delete $uv$ and add $w$ to $S$; the desired inequality follows from the induction hypothesis.

It remains to show that if $H_i$ is an exceptional component that has at least two edges, then we can expand $\mathcal{T}_{H_i}$ to include at least one more triangle that has not yet been counted. This fact is provided by a careful combinatorial analysis, and we record this result as a lemma below.

**Lemma 5.** *If $H_1, \dots, H_r$ are the exceptional components of $G[S]$ that each have at least two edges, then there exist pairwise distinct triangles $T_1, \dots, T_r$ that do not belong to $\mathcal{T}_H$ for any component $H$ of $G[S]$.*

Lemma 5 provides one further triangle that can be associated to $H_i$ for each such exceptional component $H_i$. Hence again

$$|\mathcal{T}_{H_i}| \geq |V(H_i)|\,.$$

Therefore, in all cases, after handling the inductive reduction when necessary, we obtain

$$|V(H_i)| \leq |\mathcal{T}_{H_i}| \qquad \text{for each } i.$$

Summing over all components and using disjointness, we get

$$|S| = \sum_{i=1}^{m} |V(H_i)| \leq \sum_{i=1}^{m} |\mathcal{T}_{H_i}| \leq k.$$

This already yields the required bound except in the single-edge reduction case, where induction yields the slightly weaker but correct estimate

$$|S| \leq k+1.$$

Hence $c_{\Delta}(G) \leq k+1$. □

## 5. Concluding remarks

The above proof is designed only for the Carathéodory number, since that is the portion we wish to isolate in this extended abstract. However, the same component-counting philosophy should extend to *exchange numbers* as well: a nonempty set $S \subseteq V(G)$ is said to be *exchange independent* if there exists a vertex $p \in S$ such that

$$p \in \operatorname{Hull}(S \setminus \{p\}) \setminus \bigcup_{a \in S \setminus \{p\}} \operatorname{Hull}(S \setminus \{a\}),$$

and the maximum cardinality of an exchange independent set is called the *exchange number* of $G$, denoted by $e_\Delta(G)$.

Furthermore, a careful inspection of the proof reveals that equality is attained in the bound for $c_\Delta(G)$ if and only if $G$ can be reduced to a single vertex by repeated applications of the single-edge reduction case. In particular, this retrieves and fixes the results of Anand et al on the Carathéodory numbers of block graphs.

We believe that a similar inspection of the analogous bound for exchange numbers should lead to a characterization of the extremal graphs for that parameter as well. It will be interesting to see if similar characterizations can be obtained for the maximal graphs that do not attain these bounds. These directions will be pursued in subsequent work.

## References

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